PROGRAM: 6
The Beauty of Symmetry

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Time Code	Audio
00:00	OPENING CREDITS
00:40	HOST They say, beauty is in the eye of the beholder.
00:45	HOST And in fact, what we consider to be beautiful in nature, art, or music often differs from culture to culture, nationality to nationality-- even generation to generation.
00:56	HOST But somehow, there seem to be constants -- commonalities in how we as human beings "see" beauty. There's a sense of balance, of order, to beautiful things. Where does that "sense" come from? And what does algebra or geometry have to do with it? Can we quantify the beauty of a butterfly?
01:15	HOST Welcome to the world of symmetry.
01:23	**HOST (V.O.)** **Crystals, snowflakes, shells, geometric shapes, flowers... decorative arts such as the tilings of the Alhambra Palace in Spain - they all share something besides what we might call 'beauty.' And that something that is implicit in the sort of regularity or self-similarity is symmetry – a symmetry we experience with satisfaction and pleasure in our everyday lives.**
01:49	**HOST (V.O.)** **Often, that which we experience as symmetry is simply the final picture. But how did that beautiful, symmetric pattern come into being? With some close and careful inspection, anyone can see that symmetry starts with a basic motif which is then manipulated in time and space.** **Mathematicians call such manipulations transformations or, though this might be confusing, we also call them symmetries, thereby identifying the outcome with the transformation that got us there.**
02:22	**HOST (V.O.)** **For example, start with a basic shape or pattern, say, the number 4. If we flip it over like it's on a mirror, we're using a kind of transformation called a reflection. And that reflection creates what we call bilateral symmetry.**
02:36	**HOST (V.O.)** **This bilateral symmetry seems fundamental to nature, we see it our bodies, in fact in the form of all sorts of living things. Take the butterfly: its folded wings are the basic pattern. Opening its wings, it**

	performs a reflection, resulting in one of nature's most beautiful examples of bilateral symmetry.
02:55	**HOST (V.O.)** **But visual symmetries are only the tip of the mathematical iceberg. In fact, there are an uncountable number of types of symmetries, some we can see and some that we can't, some having to do with galaxies and others having to do with subatomic particles, some having to do with magic and others having to do with equations. But in the beautiful visual symmetries that nature and art present, we see everything that we need to know…**
03:24	**HOST (V.O.)** **Symmetry is as much about the final pattern, as it is about the motions that got us to that final pattern. And it's the math of those motions that we're now interested in.**
03:37	HOST Remember - we said an object or picture is bilaterally symmetric if we can draw a line through it, what we would call an axis of symmetry, and the two halves are identical - in the sense that one half can be overlaid on the other perfectly, with each point finding its exact copy on the other side - each little fleck and curlicue matched up exactly when I fold the one onto the other.
04:01	**HOST (V.O.)** **The symmetry that the butterfly reveals and illustrates in opening up her wings is an example of a mathematical and geometric transformation that can be used to both generate and characterize symmetry - it's called a reflection, for the way in which it creates an image that would appear by reflecting the original in a mirror.** **Reflections take place outside the plane of the figure. See how we needed to actually flip our little icon or motif outside the page in order to make the mirror image? Now, there are also symmetries and transformations that take place entirely in the plane of the image.**
04:37	**HOST (V.O.)** **The simplest of these is a translation. Here we draw an imaginary line above or below our motif and simply slide cross that line without rotating or reflecting it. Notice that a translation will be completely described by how far you move and in what direction you move it.**
04:55	**HOST (V.O.)** **Another different example is given by a rotation. Here, notice that the rotation is characterized by how much you rotate as well the point around which you rotate. The closer the center of rotation, the tighter the curve that the icon traces out. Now with these three kinds of symmetries, reflections, translations, and rotations we can generate other kinds of symmetries too.**
05:18	HOST A famous one is the glide reflection; a symmetry that combines a reflection

	and a translation. A flip and slide equal a glide.
05:28	HOST Now, using these symmetries, we can form all kinds of patterns. Let’s take a basic motif - we’ve been using ‘R’ so let’s stay with it for a moment:
05:37	**HOST (V.O.)** **Start with a reflection, then add a rotation, and then maybe a glide...** **If we follow one symmetry by another symmetry, that is ‘compose’ one symmetry with another, we get another symmetry. And what we end up with can be, well, beautiful.**
05:53	**HOST (V.O.)** **We can compose symmetries along one dimension and end up with a frieze pattern. We can even expand the symmetries to cover an entire wall, and if we allowed ourselves to go out to infinity - and that's something that mathematicians love to do – we can cover the entire two-dimensional plane with a beautiful and highly symmetric pattern.**
06:14	HOST Now as we said, symmetries are both the motions that generate a beautiful pattern like this as well as the properties of some patterns. When we look at this mesmerizing infinite design, we now see that there are certain transformations in our bag of symmetries, that when applied to the design bring it right back on itself, like that. So there’s a set of symmetry transformations, that leave the entire infinite design unchanged, or “invariant” is what we say. We call this set the symmetry group of the design.
06:48	HOST Now the important word here is "group" - a group is a set of symmetries that behave sort of like the good old integers. Now what I mean by that is I can take two numbers and add them and get another number. Similarly, I can take two symmetries, apply them one after another and get another symmetry. I can take a number and find it's negative and when I add those 2 numbers I get 0. And similarly, I can take a symmetry do its inverse symmetry and it's as if I did nothing at all! Those simple properties define what we mean by a group.
07:23	HOST One of the most beautiful results in all of mathematics is the characterization or classification of the various possible groups of symmetries of these infinite planar or linear designs that we’ve been looking at. Now the former are called wallpaper groups, speaking to these infinite pieces of wallpaper that we’ve been creating, and the latter are called frieze groups. And while you might think that there are an infinite number of possibilities for these, the fact is that their structure is really highly constrained and that effectively, there are only 7 frieze groups and 17 wallpaper groups. Now, as we’ve seen, although these groups are finite in number, their possibilities for creating beauty seem endless and in fact, they've been

	inspiring magnificent geometric artistry for centuries.
08:18	**HOST (V.O.)** **This is the Alhambra, a combined mosque, palace and fortress built in Southern Spain in the 13th century. In many ways, it stands as a monument to the ways in which geometry and algebra were combined by Islamic artists as a means of spiritual expression.**
08:36	**HOST (V.O.)** **The ancient religious commandment, "Thou shall not carve idols for yourselves in the shape of anything in the sky above or on the earth below or in the waters beneath the earth" was an injunction against figurative art taken seriously by both Arabs and Hebrews. As a result, they developed a purely** **abstract and geometric art like that exemplified in the Alhambra.**
09:00	**HOST (V.O.)** **Much of the mathematics we study today owes at least as much to early Islamic mathematicians as it does to the ancient Greeks. In fact, many of the ideas developed by European Renaissance mathematicians were first developed by Islamic mathematicians four centuries earlier.**
09:18	**HOST (V.O.)** **Beginning in the late 8th century, at the House of Wisdom in Baghdad, Islamic mathematicians translated the Greek texts -- like Euclid's Elements -- into Arabic.**
09:27	**HOST (V.O.)** **Islamic mathematicians were interested in both pure and applied mathematics. They used it in astronomy, geography, time-keeping, and even in the legal arena where they used it to settle the problems of division of inheritances.**
09:41	**HOST (V.O.)** **After the work of al-Khwarizmi in the 9th century, algebra became a unifying theory which allowed things like rational numbers, irrational numbers, and geometric magnitudes, to all be treated as "algebraic objects" — that is, abstract symbols that stood in for specific concepts. It is from al-Khwarhizmi that we get the word "algorithm", a mathematical process followed by a computer.**
10:07	**HOST (V.O.)** **Omar Khayyam, born in 1048 and mostly known as a poet, was also a mathematician -- well aware of the power of combining algebra and geometry.**
10:18	**HOST (V.O.)** **Khayyam wrote that "Whosoever thinks algebra and geometry are different has thought in vain. Algebras are geometries which are proved."**
10:26	**HOST (V.O.)** **Which brings us back to the Alhambra, with its surfaces covered in repeating geometric motifs, beautifully integrated with poetry**

	carved into the walls.
10:36	**HOST (V.O.)** **"You will say upon seeing it: it is a fortress and, at the same time, a mansion of happiness. It is a dwelling for the peaceful and the warrior. It is an artistic work that produces wisdom."**
10:51	HOST Whether the Islamic designers of the Alhambra understood the mathematics of symmetry and how is a matter of conjecture. What we do know is that the work of Islamic mathematicians in spatial symmetry and algebra foreshadowed the creation of group theory -- a core concept of symmetry.
11:13	**HOST (V.O.)** **The study of symmetry allows us to look at geometric things using algebraic tools. And that lets us both solve difficult problems and connect things that don't seem to have anything to do with each other. How? It's all about the symmetry groups and something called invariance...**
11:36	Dan Rockmore: So symmetry, group theory, the math of beauty actually is what it is, is ultimately a lot of algebra, and so we're lucky to have with us today Rosa Orellana, professor of mathematics at Dartmouth College, who uses algebra all over in her work. So, Rosa, thanks for coming today.
11:47	Rosa Orellana: Thank you for having me, Dan.
11:49	Rockmore: Well, we've been seeing a lot of beautiful geometric objects and trying to make sense of them algebraically, but let's sort of make it a little bit more rigorous with some good examples, shall we?
11:59	Orellana: We should start with, like, a rectangle and try to explain the ideas because it's easy to see it there. So actually, let's have a square.
12:08	Rockmore: Okay, squares-- squares are good because squares are highly symmetric.
12:12	Orellana: So now the idea, Dan, is that we would want to have a motion in a space that brings us back to the exact configuration that we have.
12:21	Rockmore: Okay.
12:22	Orellana: So, for instance, here is the square. You close your eyes, and I do something to it, and you open it and you tell me if I did anything.
12:31	Rockmore: Okay.
12:32	Orellana: I can rotate it by 90 degrees, and this brings me actually back to the same configuration.

12:38	Rockmore: That's right, that's right. Occupies the same space and looks as if I haven't moved space at all, in fact.
12:42	Orellana: Exactly. I can do 180 degrees.
12:45	Orellana: If you think about it, 90, 180, 360 --
12:50	Orellana: --you can think 180 is 90+90. So if I do 90 and then I do 90 again, it's the same as if I would have rotated by 180 degrees, right?
13:02	Rockmore: That's right, that's right. So you're allowed to compose these moves is what we would say.
13:05	Orellana: Yeah.
13:06	Rockmore: There's a composition.
13:07	Orellana: Exactly. So this is kind of like when you are doing algebra, you start to realize, you know, "can I combine these motions so that I get a similar motion?" And in this case, we want a motion that leaves the square invariant, the same, right?
13:23	Rockmore: Yeah, and it's interesting to look at -- I mean, in fact, the square specifies that there are only particular transformations that are going to leave it alone. Like if I had rotated that thing by 30 degrees, or you did that, I would know you had moved it.
13:36	Rockmore: Okay, so now the square is of course very symmetric, we've said, but you know, you can kind of reduce its symmetry, right? I mean, reduce the symmetry of a quadrilateral…
13:47	Orellana: So if you, for instance, colored the top half yellow and the bottom half blue, like now we can ask the same question: what motion in space will leave it invariant? So that somebody wouldn't be able to tell that we have moved. And now we notice that for instance if we do like a reflection, top to bottom, right, you can tell that I moved it because the blue went to the top and the yellow went to the bottom.
14:15	Rockmore: Of course, of course.
14:16	Orellana: But so now the only possible choice seems to be that we can reflect it from left to right.
14:24	Rockmore: And this connection of numbers and symmetries is a real one, right?
14:27	Orellana:

	Exactly. Because in mathematics, one thing that we love to do is abstract from what we know…
14:34	Orellana: …here we have the numbers and this forms a set. So it's --
14:38	Rockmore: So just a collection of entities, right, of things.
14:41	Orellana: You see, you're combing two things that are the same in the same set and making a new one, right? If you take an integer, like let's take -4, and you add it to 2, we get -2, which is again in the set, and if you add 2+4, you get 6.
15:04	Rockmore: Right, so always getting some other thing on the line there.
15:07	Orellana: So this is an example of a binary operation, binary meaning "two". You take two objects, you combine them, and you get a new object that is back in the set. And this we call closure…Like the operation doesn't take you away from the set.
15:22	Rockmore: Okay, so we have our binary operation, and it makes sense for this collection of integers in the sense that if I take two of them and do addition, I get another one, right?
15:31	Orellana: Exactly. So now you might ask, what other properties does this operation have? So let's say I give you 1, 2, and 3, and I ask you add them, give me the result. If somebody's new to it, will go 1+2 and then add the third number to that result of 1+2 or go 2+3 and add that result to the 1.
15:57	Rockmore: Right, to 1, right.
15:58	Orellana: So and the question is do we get the same answer when we do it in these two ways, right? And this is the property that we call associativity.
16:09	Orellana: So another thing that is quite an amazing thing is there's an element such that when I combine it with any other element, it doesn't change the value?
16:19	Rockmore: Right, right.
16:20	Orellana: Like, you know, is there an identity element, something that doesn't change?
16:24	Rockmore: … an identity element, okay.
16:25	Orellana: Yes, an identity element. If we add zero to any other integer, we get whatever integer we obtain. So for instance, if you get four and you add it to

	zero, you get --
16:38	Rockmore: You still have four.
16:39	Orellana: -- four. And it doesn't --
16:40	Rockmore: So four -- so four retains its identity.
16:32	Orellana: So -- and even if we do it in either way, 4+0 or 0+4, you still get 4, and this is the property of having an identity.
16:52	Orellana: So now that you have an identity element, you might ask yourself, if I give you an element in the set, is there an element that would give us zero? So for instance, if I give you 3, you might come up and say, you add -3 and you get zero.
17:10	Rockmore: Right, so every element on the line has its negative, and those are sort of specified by the fact that when you combine those two, you get this identity.
17:17	Orellana: This identity element, and this is the existence of inverses.
17:22	Rockmore: Of inverse…in this case an additive inverse.
17:23	Orellana: An additive inverse, exactly, for the integers. So these four axioms have been recognized as being special. You start with a set, you add closure, associativity, you have this identity element, and you have this inverse for every element in the set. And these four properties is what forms a group. This is, you know, is a set with these four properties.
17:48	Rockmore: And it's sort of amazing that out of those four axioms somehow, that you get an incredible richness, you know, from the wallpaper groups that we saw to, you know, groups in physics …
17:57	Rockmore: We've been talking about groups in a very geometric way: moving designs around, moving shapes around. But in fact, the birth of group theory is-- was much more formal, right, sort of closer to the algebra that people think of being algebra, isn't that right?
18:10	Orellana: Yeah. Usually what people think about when you say algebra, they think, you know, high school … but the first person to use the term "group" was Evariste Galois.
18:21	Rockmore: Ah, very famous mathematician, and we're going to hear his fascinating story right now. Let's take a look.
18:26	Orellana: Let's take a look.

18:28	**HOST (V.O.)** **A firebrand radical Republican who lived in the time of King Louis Philippe -- Galois was just twenty years old when he solved one the most hotly contested problems in mathematics: The problem of solvability by radicals. And it goes like this:** **"Under what conditions could you find a formula for the roots of any polynomial in terms of the coefficients using only the usual algebraic operations and application of radicals, square roots, cube roots, and so on.**
18:57	**HOST (V.O.)** **Galois's quest was a powerful result: to be able to say for any polynomial with integer coefficients, both when it was possible and impossible, to find one of these simple formulas for its roots in terms of the coefficients.**
19:11	**HOST (V.O.)** **Galois' epiphany was to consider the symmetries of the roots of the polynomial. The young Frenchman's answer became known as Galois Theory, and is the basis of modern group theory and our understanding of the mathematics of symmetry.**
19:26	**HOST (V.O.)** **On May 29, 1832, Galois is said to have worked through the night desperately composing his mathematical treatise summarizing six years of work. It was the night before he would be fatally wounded in a duel.**
19:40	**HOST (V.O.)** **With apparent foreboding, Galois wrote: "I have no time."**
19:46	**HOST (V.O.)** **While the circumstances remain murky -- the duel may have been the result of romantic entanglement or political intrigue -- Galois' legacy is the algebraic basis of our understanding of symmetry.**
20:04	HOST So, Galois discovered a set of conditions in terms of the symmetries of the roots of the polynomial that could determine if the polynomial could be solved by radicals!
20:13	HOST But the roots are numbers, and what could it mean to consider the symmetries of a collection of numbers? Well, let's take a step back, all the way the back to the simple example of the quadratic formula. Really, the quadratic formula tells us two things - the familiar one is that from just the coefficients of the polynomial I can write down the roots. But it also works in the other direction: knowing the roots, I can figure out the coefficients of a polynomial that they turn into zero!
20:41	HOST So, Galois's idea was exactly to adapt this backward point of view - and add a very clever twist: he would use these roots as the basic ingredients to start

	making numbers. He'd take the roots, toss in all the rational numbers too and then see what numbers he could make taking arbitrary sums and products. Like taking a pile of atoms and seeing what sorts of molecules you can make. Decades later, scientists would begin to understand the properties of molecules by determining their invariance under symmetries that mix up the atoms. In a similar way, Galois discovered that the ability to solve a polynomial built out of some roots depended on the invariance of those numbers created from the roots after permuting the roots.
21:26	HOST Permuting - it means mixing up or "shuffling" and in order to see what we have is a group, let's actually think about shuffling, but card-shuffling. Here's a familiar old deck of cards - and notice that any shuffle, or rearrangement of the deck leaves it invariant. In our new language, it's a symmetry of the deck of cards. Notice that if I do two shuffles in a row, I get another shuffle - so closure. Notice that for every shuffle there's an inverse shuffle – just the one that undoes the shuffle. Not doing anything at all gives me the identity shuffle and take my word, associativity works!
22:10	HOST So the shuffles of a deck of cards are a group - and it turns out that the structure of the particular "shuffles" that Galois considered on his roots are what give the conditions for solvability by radicals. Using shuffles to understand polynomials is a pretty good trick, but an even better one is using the algebra of shuffles to actually understand card shuffling – which is something that people have been doing for decades, culminating the recent mathematical discovery that in order to mix a deck of cards by riffle shuffling - that is it takes seven riffle shuffles to randomize an ordered deck of cards. This was proved in 1992 by Persi Diaconis and David Bayer.
22:56	HOST Diaconis and Bayer identified a key signature of a shuffled deck of cards - the number of rising sequences. Let's look at an example. I start with an ordered deck of 10 cards, riffle shuffle them once and take a look. See how it's now composed of 2 rising sequences, 2 sub sequences of increasing cards, and as I continue to shuffle the order dissolves. After I do another shuffle, I expect 4 rising sequences, and then after that 8 and so on and so forth. The way in which permutations of a given number of rising sequences compose - their algebra – provides the key to understanding how the mathematical model of riffle shuffling works.
23:38	HOST This is just one example of what is in fact a very intriguing intersection of the world of cards, and even card tricks, with mathematics. But here's another trick - did you know that we can use symmetry to uncover virtually invisible worlds of atoms in a crystal? Well, we can - let's take a look!
23:57	HOST So crystallography is the study of how atoms or molecules are arranged in the solid form.
24:04	McGRATH

	My name is Mary McGrath. I'm the Senior Director of structural chemistry here at Gilead Sciences.
24:12	McGRATH Our goal at Gilead is to get safe and effective drugs to patients sooner.
24:22	McGRATH So in the field of crystallography, symmetry is very important. Symmetry tells you how each molecule in the crystal is related to the other molecules. So we need to know that information in order to get out our final picture of what our protein looks like. It's also very important in just trying to complete our experiments successfully. We need to know those relationships.
24:55	McGRATH What we end up doing is using math as a lens to refocus our image.
25:01	McGRATH When we are able to provide a picture in crystallography, of the protein target to the chemists, they are able to specifically design for that protein. So what ends up happening is you accelerate the drug discovery process. Instead of taking ten years to come up with a potent compound, maybe now it would take three or four years.
25:25	McGRATH Our process has many parts to it and one of the things I really love about being a protein crystallographer and doing structure guided drug design is that we use techniques that have really not changed all that much for close to 100 years and we use other technology that's just changing all the time. So it's a really fun blend of the traditional and the cutting edge.
25:54	**HOST (V.O.)** **The uncovering of symmetries in a crystal are just one example of the use of symmetry in physics. A much more mysterious fact is that laws of nature generally exhibit some** **sort of invariance, a fact proved by Emmy Noether, perhaps the greatest woman mathematician of the twentieth century. A prime example is the law of conservation of** **momentum that comes from the spatial invariance of the laws of motion. Fundamentally, the world works via symmetry.**
26:28	HOST The symmetries of nature, the patterns of the Alhambra tilings, the manipulations of mathematical equations or a deck of cards…whether we're talking about symmetry as a property or as a transformation, it creates the emotional responses we "feel" in the presence of beauty, whether it's art, math... or butterflies.
26:50	CREDITS